

Singularity in Kerr-Newman spacetimes endowed with negative mass

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The Kerr-Newman solution with negative mass is shown to develop a massless ring singularity *off* the symmetry axis. The singularity is located inside the region with closed timelike curves which has topology of a torus and lies outside the ergoregion.

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I. INTRODUCTION

It is well known that the Kerr-Newman (K-N) solution [1] contains three arbitrary real parameters m , a and q , which denote, respectively, the mass, angular momentum per unit mass and charge of the source, and describes the exterior field of a charged rotating black hole when m is positive and satisfies the inequality $m^2 \geq a^2 + q^2$. The Ernst complex potentials [2] of this solution can be constructed from the axis data

$$e(z) \equiv \mathcal{E}(\rho = 0, z) = \frac{z - m - ia}{z + m - ia}, \quad f(z) \equiv \Phi(\rho = 0, z) = \frac{q}{z + m - ia}, \quad (1)$$

with the aid of Sibgatullin's integral method [3], yielding the expressions [4, 5]

$$\mathcal{E} = \frac{\kappa x - m - iay}{\kappa x + m - iay}, \quad \Phi = \frac{q}{\kappa x + m - iay}, \quad \kappa = \sqrt{m^2 - a^2 - q^2}, \quad (2)$$

written in the generalized spheroidal coordinates (x, y) which are related to the Weyl-Papapetrou cylindrical coordinates (ρ, z) by the formulas

$$x = \frac{1}{2\kappa}(r_+ + r_-), \quad y = \frac{1}{2\kappa}(r_+ - r_-), \quad r_{\pm} = \sqrt{\rho^2 + (z \pm \kappa)^2}. \quad (3)$$

To rewrite \mathcal{E} and Φ in the “standard” Boyer-Lindquist coordinates (r, θ) , one has to perform in (2) the substitution

$$\kappa x = r - m, \quad y = \cos \theta, \quad (4)$$

thus obtaining

$$\mathcal{E} = \frac{r - ia \cos \theta - 2m}{r - ia \cos \theta}, \quad \Phi = \frac{q}{r - ia \cos \theta}. \quad (5)$$

It follows then that the singularity of the K-N solution (zero of the denominator of \mathcal{E}) occurs when

$$r = 0, \quad \cos \theta = 0, \quad (6)$$

independently of the sign of the mass parameter m . However, from (4) one can easily see that whereas the introduction of the Boyer-Lindquist coordinates in the case $m > 0$ leads to an extension ($r > \kappa x$) of the K-N solution and hence is mathematically justified, the transformation (4) in the case $m < 0$ causes a contraction ($r < \kappa x$) of the K-N spacetime and is, therefore, inappropriate for a correct interpretation of the K-N solution with negative mass.

The main objective of the present paper is to show that the singularity of the negative-mass K-N metric is a massless ring singularity which is located off the symmetry axis, inside the region with closed timelike curves (CTCs). In the next section all necessary formulas for performing the required analysis will be given and location of the singularity will be established. In Sec. III several analytical results concerning the ring singularity, ergoregion and region with CTCs are obtained, and two particular examples are considered. Concluding remarks are given in Sec. IV.

II. THE K-N METRIC AND ITS RING SINGULARITY IN THE $m < 0$ CASE

As the Boyer-Lindquist coordinates are problematic for treating the case of negative mass, it is advantageous for our purpose to use a representation of the K-N metric in generalized spheroidal coordinates which is defined by the

line element

$$ds^2 = \kappa^2 f^{-1} \left[e^{2\gamma} (x^2 - y^2) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\varphi^2 \right] - f(dt - \omega d\varphi)^2, \quad (7)$$

with the following metric coefficients f , γ and ω [4]:

$$\begin{aligned} f &= \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{(\kappa x + m)^2 + a^2 y^2}, \quad e^{2\gamma} = \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{\kappa^2(x^2 - y^2)}, \\ \omega &= -\frac{a(1 - y^2)[2m(\kappa x + m) - q^2]}{\kappa^2(x^2 - 1) - a^2(1 - y^2)}. \end{aligned} \quad (8)$$

Formulas (8) permit one to study in a unified manner the cases of subextreme (real κ) and hyperextreme (pure imaginary κ) K-N single sources possessing a positive or negative mass. Furthermore, the norm $\eta^\alpha \eta_\alpha$ of the axial Killing vector permitting one to analyze the region with CTCs can be shown to have the form

$$\begin{aligned} \eta^\alpha \eta_\alpha &= \kappa^2(x^2 - 1)(1 - y^2)f^{-1} - f\omega^2 = \frac{(1 - y^2)\mathcal{N}}{(\kappa x + m)^2 + a^2 y^2}, \\ \mathcal{N} &= [(\kappa x + m)^2 + a^2]^2 - \kappa^2 a^2 (x^2 - 1)(1 - y^2), \end{aligned} \quad (9)$$

while for the only non-zero components of the Weyl and Maxwell tensors of the K-N solution we have the expressions [6]

$$\Psi_2 = -\frac{m(\kappa x + m + iay) - q^2}{(\kappa x + m - iay)^3(\kappa x + m + iay)}, \quad \sqrt{\pi}\Phi_1 = \frac{q}{4(\kappa x + m - iay)^2}, \quad (10)$$

which show that the singularity of the K-N metric coincides with the singularity of the potential \mathcal{E} in (2), and hence its location is the following:

$$\kappa x + m = 0, \quad y = 0. \quad (11)$$

In the subextreme black-hole case ($m > 0$, $m^2 > a^2 + q^2$), extensively discussed by Carter [7], extension of the coordinate x to negative values is needed for attaining the singularity because both m and κ in this case are positive definite, as well as x by virtue of (3), while the solution of the first equation in (11) is $x = -m/\kappa < 0$. The introduction of the Boyer-Lindquist coordinates via (4) then implements a “standard” extension of the solution leading to (6) instead of (11), the singularity being surrounded by the event horizon $r = m + \kappa$ (the hypersurface $x = 1$).

When $m < 0$, $m^2 > a^2 + q^2$, the first equation in (11) has one positive root

$$x = -\frac{m}{\kappa} = -\frac{m}{\sqrt{m^2 - a^2 - q^2}} > 1, \quad (12)$$

and consequently no special extension of coordinates is needed for attaining the corresponding singularity since (12) defines, together with $y = 0$, a ring located *outside* the symmetry axis. Indeed, $y = 0$ is equivalent to $z = 0$ (the equatorial plane), and then (12) with the aid of (3) can be trivially solved for ρ , yielding instead of (11)

$$\rho = \sqrt{a^2 + q^2}, \quad z = 0. \quad (13)$$

Obviously, (13) describes a ring of radius $\sqrt{a^2 + q^2}$ lying in the equatorial plane and having its center at the symmetry axis ($\rho = 0$).

Remarkably, in the case of hyperextreme or extreme K-N sources with negative mass ($m < 0$, $m^2 \leq a^2 + q^2$) the ring singularity is again described by formulas (13), which can be readily verified by rewriting $\kappa x = -m$, $y = 0$ in cylindrical coordinates.

In the absence of rotation ($a = 0$) the K-N solution reduces to the Reissner-Nordström spherically symmetric spacetime [8, 9], and in the case of negative mass, the K-N singular ring converts into a singular sphere which in the ρ and z coordinates emerges as a spheroid with the equatorial radius $|q|$ and the poles at $z = \mp m$. Formally this is due to the fact that the denominators in (2) and (10) become independent of y when $a = 0$, but intrinsically the reason for a new shape of the singularity is a symmetry change. Mention also that in the particular case of the Schwarzschild spacetime with $m < 0$, the whole hypersurface $x = 1$ becomes singular.

It is not difficult to show that the singularity of the K-N solution with $m < 0$ is *massless*, so that the whole negative mass comes from the segment $x = 1$ ($\rho = 0$, $|z| \leq \kappa$) of the z -axis. Indeed, calculating the Komar mass M [10] of the latter segment with the aid of Tomimatsu's formula [11]

$$M = -\frac{1}{4}\omega_0[\Omega(x=1, y=1) - \Omega(x=1, y=-1)], \quad (14)$$

where ω_0 is the value of ω on the hypersurface $x = 1$ and Ω denotes the imaginary part of the potential \mathcal{E} , namely,

$$\Omega = -\frac{2may}{(\kappa x + m)^2 + a^2 y^2}, \quad (15)$$

one readily arrives at the value m that coincides with the total mass of the K-N source obtainable from the asymptotic expansion of the metric coefficient f .

III. ERGOSURFACE AND REGION WITH CTCs

The ergosurface, also known as infinite redshift surface, is defined by the equation $f = 0$, or explicitly

$$\kappa^2(x^2 - 1) - a^2(1 - y^2) = 0, \quad (16)$$

and its shape does not depend on the sign of the mass parameter m . It is well known that in the subextreme case its topology is of a sphere, and in the extreme and hyperextreme cases – that of a torus. So in what follows we can restrict our consideration to only the subextreme case and occupy ourselves with answering the most interesting question about this surface: what is its location relative to the naked singularity in the case $m < 0$?

To answer this question, we must consider the S^1 intersection of the ergosurface with the equatorial plane, i.e., we have to put $y = 0$ in (16) and solve the resulting equation for x . The positive root then has the form

$$x = \frac{\sqrt{m^2 - q^2}}{\kappa}, \quad (17)$$

and its value is less than $x = -m/\kappa$ defining the ring singularity, which means that the singularity is located *outside* the ergosurface and ergoregion. In the cylindrical coordinates the above intersection is given by

$$\rho = |a|, \quad z = 0, \quad (18)$$

and we see again that this is closer to the symmetry axis than the location of the ring singularity $\rho = \sqrt{a^2 + q^2}$, $z = 0$. Only in the absence of charge ($q = 0$) the two rings coincide, the singularity then locating at the equator of the ergosurface.

Another region of interest emerging in the K-N spacetime with negative mass is the region with CTCs where the norm of the axial Killing vector (9) takes negative values and causality violation occurs. The boundary of this region is defined by the equation

$$\mathcal{N} = 0, \quad (19)$$

while the region itself consists of the points for which $\mathcal{N} < 0$.

The analytical study of (19) is more difficult than in the case of the Kerr metric [12] with negative mass, and the aggravation is clearly due to the presence of electromagnetic field, but of course a numerical analysis of (19) does not represent any difficulty. Nonetheless, it is still possible to give analytic proof to the following three general statements ($m < 0$, $a \neq 0$, $q \neq 0$):

- (i) the boundary of the region with CTCs has no common points with the ergosurface;
- (ii) the ring singularity belongs to the region with CTCs;
- (iii) the region with CTCs lies entirely outside the ergoregion.

To prove (i), it is sufficient to consider a linear combination of Eq. (19) with Eq. (16) multiplied by $a^2(1 - y^2)$, thus yielding

$$[(\kappa x + m)^2 + a^2 y^2][(\kappa x + m)^2 + a^2(2 - y^2)] = 0. \quad (20)$$

The second factor in (20) is always positive definite since $y^2 \leq 1$, while the first factor vanishes at $x = -m/\kappa$, $y = 0$, i.e., at the singularity. The substitution of the latter values into (16) then leads to the condition $q^2 = 0$, which contradicts the initial assumption that q is non-vanishing. This proves (i).

To verify (ii), it is only necessary to substitute the values of x and y defining the singularity into the expression for \mathcal{N} , getting

$$\mathcal{N}(y = 0, x = -m/\kappa) = -a^2 q^2 < 0, \quad (21)$$

whence it follows that the ring singularity does belong to the region with CTCs.

Lastly, since the ring singularity, as has already been established, lies outside the ergoregion and belongs to the region with CTCs whose boundary has no common points with the ergosurface, then the latter region lies entirely outside the ergoregion.

The extent in the equatorial plane of the region with CTCs is determined by two real roots of the quartic equation

$$\mathcal{N}(y = 0, x) = [(\kappa x + m)^2 + a^2]^2 - \kappa^2 a^2 (x^2 - 1) = 0, \quad (22)$$

which, however, are not given here because of their cumbersome explicit form. Note that in the vacuum limit ($q = 0$), Eq. (22) admits partial factorization due to a common ring singularity shared by the ergosurface and boundary of the CTC region and, as a consequence, the corresponding roots have a rather simple form [13].

In Figs. 1 and 2 the ergoregion, region with CTCs and ring singularity are plotted for two particular choices of the parameters. Fig. 1(a) represents a typical subextreme case of the K-N solution with negative mass ($m = -3$, $a = 2$, $q = 1$), and though it might look that the ergosurface touches the boundary of the region with CTCs, this is not really so, as can be well seen from Fig. 1(b), but the two surfaces approach each other very closely. The three positive values of ρ defining the points at which the equatorial plane intersects the ergosurface and boundary of the region with CTCs are the following: 2, 2.013, 5.122, while the location of the ring singularity is defined by $\rho \approx 2.236$. Fig. 2 describes the particular hyperextreme case $m = -2$, $a = 3$, $q = 1$, for which there are four intersections of the equatorial plane with the ergosurface and boundary of the region with CTCs defined by the positive values of ρ : 2, 3, 3.019, 5.167, while the ring singularity is located at $\rho \approx 3.162$. Note that if in the subextreme case only the region with CTCs has toroidal topology, in the hyperextreme case both the ergoregion and region with CTCs have topology of a torus.

IV. CONCLUDING REMARKS

The present paper may be considered as a useful complement to the known positive mass theorems [14–16] for black holes. It clearly demonstrates that the negative mass in the stationary axisymmetric K-N spacetime is a source of serious pathologies – a massless ring singularity and region with CTCs, both of which emerge outside the symmetry axis. It also attracts attention to an interesting fact that the Boyer-Lindquist coordinates are too restrictive to describe correctly the singularity structure of the K-N solution in the negative-mass case. Finally, we expect that the present work will stimulate investigation of singularities in the configurations composed of several Kerr and K-N sources by analytical means.

Acknowledgments

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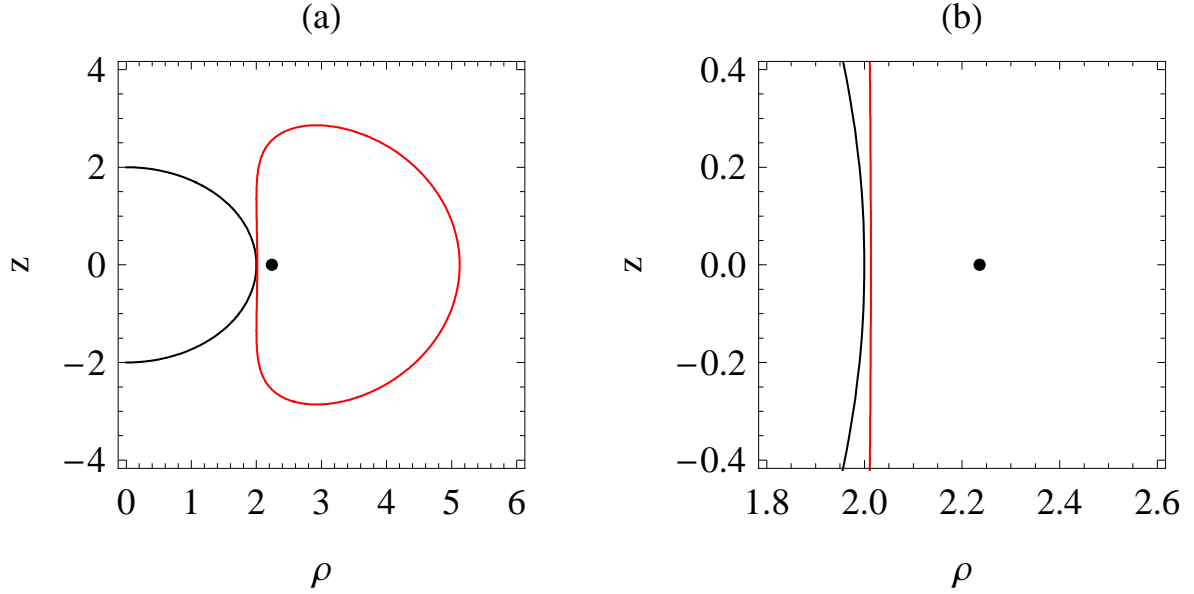


FIG. 1: In Fig. 1(a) the ergosurface (black curve), region with CTCs (inside the red curve) and ring singularity (black dot) of the K-N subextreme solution with negative mass are plotted. Fig. 1(b) presents a fragment of the plot 1(a) for illustrating that the ergoregion and region with CTCs do not touch each other.

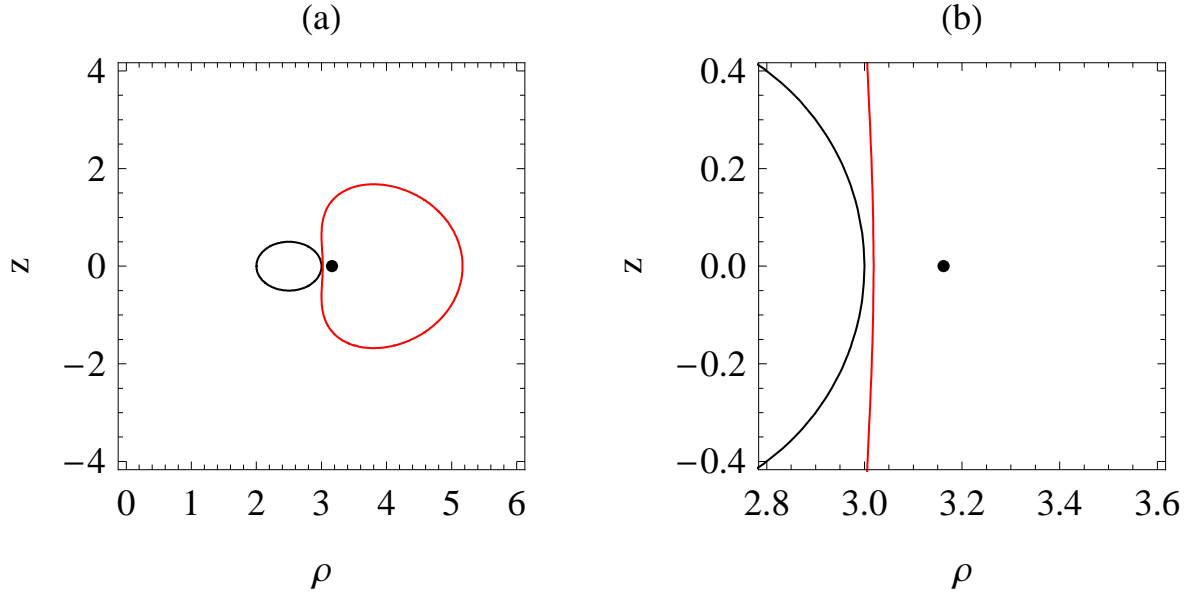


FIG. 2: In Fig. 2(a) the ergoregion (inside the black curve), region with CTCs (inside the red curve) and ring singularity (black dot) of the K-N hyperextreme solution with negative mass are plotted. Fig. 2(b) presents a fragment of the plot 2(a) for illustrating that the ergoregion and region with CTCs do not touch each other in the hyperextreme case too.